Quotient-3 Cordial Labelling For Cycle Related Graphs

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Abstract: Let $G$ be a graph of order $p$ and size $q$. Let $f: V(G) \to \mathbb{Z}_3 - \{0\}$ be a function. For each $E(G)$ define $f^*: E(G) \to \mathbb{Z}_3$ by $f^*(uv) = \left\lfloor \frac{f(u) + f(v)}{3} \right\rfloor$ (mod 3) where $f(u) \geq f(v)$. If the number of vertices having label $i$ and the number of vertices having label $j$ differ by maximum 1, the number of edges having label $k$ and the number of edges having label $l$ differ by maximum 1 then the function $f$ is said to be quotient-3 cordial labeling of $G$. $1 \leq i, j \leq 3$, $0 \leq k, l \leq 2$. Here we proved that $C_n$ and some cycle related graphs like $[C_n; C_3]$, $n=3,6,9,\ldots$, $(P_2 U m_k) + N_2$, $S(C_n; S_2)$, joint sum of $C_n$ and two cycles $C_n$ having a common vertex is quotient-3 cordial.

Key words: cycle, joint sum, subdivision, quotient-3 cordial graph.

1. Introduction

Here the graphs considered are finite, simple, undirected and non trivial. Graph theory has a good development in the graph labeling and has a broad range of applications. Refer Gallian [2] for more information. The cordial labeling concept was first introduced by cahit [1]. The quotient-3 cordial labeling have been introduced by P. Sumathi, A. Mahalakshmi and A. Rathi found in [5-7]. They found some family of graphs are quotient-3 cordial. For notations and terminology we follow [8]. If $G$ receives quotient-3 cordial labeling then $G$ is called as quotient-3 cordial graph. The number of vertices having label $i$ denotes $v_i$, and the number of edges having label $k$ denotes $e_k$, $1 \leq i \leq 3$, $0 \leq k \leq 2$.

2. Preliminaries

Definition: 2.1 $[C_n; C_3]$ graph is obtained by attaching the cycle $C_3$ with every vertex of $C_n$.

Definition: 2.2 $(P_2 U m_k) + N_2$ graph is obtained with the vertex set $V = \{z_1, z_2, x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2\}$ and the edge set $\{(y_1, z_1), (y_1, z_2), (y_2, z_1), (y_2, z_2), (z_1, x_1), (z_1, x_2), (z_2, x_1), (z_2, x_2) : 1 \leq i \leq m\}$.

Definition: 2.3 A graph $(C_n; S_2)$ is obtained by attaching the star $S_2$ with each vertex of a cycle $C_n$ through an edge.
Definition: 2.4 From G, a graph $S(G)$ is obtained by subdividing every edge of G with a new vertex called subdivision of G.

Definition: 2.5
A vertex of first copy of $C_n$ is connected with a vertex of second copy of $C_n$ through an edge is said to be joint sum of $C_n$.

3. Main Result

Definition: Let G be a graph of order p and size q. Let $f: V(G) \rightarrow \mathbb{Z}_4 \setminus \{0\}$ be a function. For each $E(G)$ define $f^*: E(G) \rightarrow \mathbb{Z}_3$ by $f^*(uv) = (f(u) + f(v)) \mod 3$ where $f(u) \geq f(v)$. If the number of vertices having label i and the number of vertices having label j differ by maximum 1, the number of edges having label k and the number of edges having label l differ by maximum 1 then the function f is said to be quotient-3 cordial labeling of G. $1 \leq i, j \leq 3, i \neq j$ and $0 \leq k, l \leq 2, k \neq l$.

Illustration: 1 A quotient-3 cordial graph

Theorem: 3.1 All cycles $C_n$ are quotient-3 cordial for $n \geq 4$ ($n \neq 9, 15, 21, \ldots$)

Proof: Let $V(G) = \{x_i : 1 \leq i \leq n\}$ and $E(G) = \{(x_ix_1), (x_ix_{i+1}) : 1 \leq i \leq n-1\}$.

Here $|V(G)| = |E(G)| = n$.

Let $f: V(G) \rightarrow \mathbb{Z}_4 \setminus \{0\}$

Case (i): when $n \equiv 0, 1, 4, 5 \pmod{6}$

For all $i$, $f(x_i) = 1$, $i \equiv 1, 2 \pmod{6}$

For all $i$, $f(x_i) = 3$, $i \equiv 0, 3 \pmod{6}$

For all $i$, $f(x_i) = 2$, $i \equiv 4, 5 \pmod{6}$

Case (ii): when $n \equiv 2 \pmod{6}$

Labeling of $x_i$, $1 \leq i \leq n-3$ are same as in case (i).

In this case, label the vertices $x_{n-2}$, $x_{n-1}$, $x_n$ by 2, 1 and 3 respectively.

<table>
<thead>
<tr>
<th>Nature of n</th>
<th>$V_f(1)$</th>
<th>$V_f(2)$</th>
<th>$V_f(3)$</th>
</tr>
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<tbody>
<tr>
<td>$n \equiv 0 \pmod{6}$</td>
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<td>$\frac{n}{3}$</td>
<td>$\frac{n}{3}$</td>
</tr>
<tr>
<td>$n \equiv 1, 4 \pmod{6}$</td>
<td>$\frac{n+2}{3}$</td>
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<td>$\frac{n+2}{3} - 1$</td>
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<td>$n \equiv 2, 5 \pmod{6}$</td>
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</tr>
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</table>
Theorem 3.2 The graph \([C_n ; C_3]\) is quotient-3 cordial for \(n = 3, 6, 9, \ldots\)

Proof: Let \(V(G) = \{(x_i, 1 \leq i \leq n) \cup \{v_{ij} : 1 \leq i \leq n, j = 1, 2\}\}\)

\(E(G) = \{(x_i,x_{i+1}) : 1 \leq i \leq n-1\} \cup \{(x_i,v_{ij}) : 1 \leq i \leq n, j = 1, 2\} \cup \{(v_{ij},v_{i+1}) : 1 \leq i \leq n\}\)

Let \(|V(G)| = 3n\), \(|E(G)| = 4n\).

Define \(f: V(G) \rightarrow \mathbb{Z}_4 - \{0\}\)

Labeling of \(x_i, 1 \leq i \leq n\) is given below.

\(f(x_i) = 1, \quad i \equiv 1, 2 \pmod{3}\)
\(f(x_i) = 3, \quad i \equiv 0 \pmod{3}\)

Labeling of \(v_{ij}, 1 \leq i \leq n, 1 \leq j \leq 2\) is given below

\(f(v_{ij}) = 1, \quad i \equiv 1 \pmod{3}, j = 1\)
\(f(v_{ij}) = 3, \quad i \equiv 1 \pmod{3}, j = 2\)
\(f(v_{ij}) = 3, \quad i \equiv 0 \pmod{3}, j = 1\)
\(f(v_{ij}) = 2, \quad i \equiv 2 \pmod{3}, j = 1, 2\)
\(f(v_{ij}) = 2, \quad i \equiv 0 \pmod{3}, j = 2\)

For all \(n\), \(v_f(1) = v_f(2) = v_f(3) = n\)

\(e_f(0) = e_f(1) = e_f(2) = \frac{4n}{3}\).

Theorem 3.3 A graph \((P_2 \cup nK_1) + N_2\) is quotient-3 cordial \((n \neq 2, 5, 8, \ldots)\)

Proof: Let \(V(G) = \{u_i, v_i, w_j : 1 \leq j \leq n, i = 1, 2\}\) and

\(E(G) = \{((u_i,u_2), (v_1,u_1), (v_1,u_2), (v_2,u_1), (v_2,u_2)) : 1 \leq j \leq n\}\)

Let \(|V(G)| = 4 + n\), \(|E(G)| = 5 + 2n\).

Let \(f: V(G) \rightarrow \mathbb{Z}_4 - \{0\}\)

\(f(u_1) = f(v_1) = 1\)
\(f(u_2) = f(v_2) = 3.\)

Label the vertices \(w_j, 1 \leq j \leq n\) as follows

\(f(w_1) = 2\)

When \(n \equiv 0, 1 \pmod{3}\)

For \(2 \leq j \leq n\),

\(f(w_j) = 1, \quad j \equiv 2 \pmod{3}\)
\(f(w_j) = 2, \quad j \equiv 0 \pmod{3}\)
\(f(w_j) = 3, \quad j \equiv 1 \pmod{3}\)

### Table 1

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<th>Nature of (n)</th>
<th>(e_f(0))</th>
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<th>(e_f(2))</th>
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</tr>
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<td>(n \equiv 2 \pmod{6})</td>
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</tr>
</tbody>
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Table 2

Theorem: 3.4 \(S(C_n;S_2)\) are quotient-3 cordial.
Proof: The cycle \(C_n\) has the vertices \(u_1, u_2, u_3, \ldots, u_n\). Let \(x_i\) be the vertices subdividing the edges \((u_iu_{i+1})\) for all \(1 \leq i \leq n-1\) and the vertex \(x_n\) subdividing the edge \(u_nu_1\).

Let \(V(G) = \{[u_i, x_i]: 1 \leq i \leq n\}\) and \(E(G) = \{[(u_1u_n), (x_nu_1), (u_iu_{i+1}), (x_iu_{i+1}): 1 \leq i \leq n-1]\}\).

Let \(f(V(G)) = |E(G)| = 6n\).

Define \(f: V(G) \rightarrow \mathbb{Z}/4\) - \{0\} by

\[
\begin{align*}
  f(u_i) &= f(x_i) = 3 \\
  f(v_{ij}) &= 1 \\
  f(w_{ij}) &= 2
\end{align*}
\]

For all \(n\), \(v_f(1) = v_f(2) = v_f(3) = 2n\)
\(e_f(0) = e_f(1) = e_f(2) = 2n\).

Theorem: 3.5 Joint sum of \(C_n\) are quotient-3 cordial.
Proof: The first cycle \(C_n\) has the vertices \(u_1, u_2, \ldots, u_n\) and another cycle \(C_n\) has the vertices \(u_{n+1}, u_{n+2}, \ldots, u_{2n}\).

Let \(V(G) = \{[u_i]: 1 \leq i \leq 2n]\}\).

\(E(G) = \{[(u_1u_n), (u_iu_{i+1}): 1 \leq i \leq n-1]\} \cup [(u_{2n+1}u_1)] \cup [(u_iu_{n+1}), (u_{n+1}u_{2n}): n+1 \leq i \leq 2n-1]\} \cup [(u_{2i+1}v_{ij}): 1 \leq i \leq n, j = 1, 2]\) \cup \{[v_jw_{ij}): 1 \leq i \leq n, j = 1, 2\}\).

Let \(|V(G)| = |E(G)| = 6n\).

Define \(f: V(G) \rightarrow \mathbb{Z}/4\) - \{0\} by

\[
\begin{align*}
  f(u_i) &= 1, \quad i \equiv 1, 2 \pmod{6} \\
  f(u_i) &= 3, \quad i \equiv 0, 3 \pmod{6} \\
  f(u_i) &= 2, \quad i \equiv 4, 5 \pmod{6}
\end{align*}
\]

\(e_f(0) = e_f(1) = e_f(2) = 2n\).

For all \(n\), \(v_f(1) = v_f(2) = v_f(3) = 2n\)
\(e_f(0) = e_f(1) = e_f(2) = 2n\).

Case (i): when \(n \equiv 0, 2, 3, 5 \pmod{6}\)
For \(1 \leq i \leq 2n\)
\(f(u_i) = 1, \quad i \equiv 1, 2 \pmod{6} \\
  f(u_i) = 3, \quad i \equiv 0, 3 \pmod{6} \\
  f(u_i) = 2, \quad i \equiv 4, 5 \pmod{6}
\)

Case (ii): when \(n \equiv 1 \pmod{6}\)
For \(1 \leq i \leq n\)
\(f(u_i) = 2, \quad i \equiv 1, 2 \pmod{6} \\
  f(u_i) = 3, \quad i \equiv 0, 3 \pmod{6} \\
  f(u_i) = 1, \quad i \equiv 4, 5 \pmod{6}
\)

For \(n+1 \leq i \leq 2n\)
f(u_i) = 3, \quad i \equiv 1, 4 \pmod{6}
f(u_i) = 1, \quad i \equiv 2, 3 \pmod{6}
f(u_i) = 2, \quad i \equiv 0, 5 \pmod{6}

In this case after labeling all the vertices interchange the label of u_{n+3} and u_{n+4}.

**Case (iii):** when n \equiv 4 \pmod{6}

Labeling of the vertices u_i, 1 \leq i \leq 2n-1 [i \neq n] are same as in Case (i)

Here f(u_n) = 3, \quad f(u_{2n}) = 2.

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</tr>
</tbody>
</table>

**Table 3**

**Theorem:** 3.6 Two copies of cycle C_n having a common vertex is quotient-3 cordial.

**Proof:** Let V(G) = \{u_i: 1 \leq i \leq 2n-1\} and

E(G) = \{(u_iu_{i+1}, u_iu_{i+1}): 1 \leq i \leq n-1\} U \{(u_1u_{n+1}), (u_1u_{2n-1}), (u_iu_{i+1}): n+1 \leq i \leq 2n-2\}

Let |V(G)| = 2n - 1 and |E(G)| = 2n

Let f: V(G) \rightarrow \mathbb{Z}_4 - \{0\}

**Case (i):** When n \equiv 0 \pmod{6}

f(u_{2n-1}) = 3.

For 1 \leq i \leq 2n-2

f(u_i) = 1, \quad i \equiv 1, 2 \pmod{6}
f(u_i) = 3, \quad i \equiv 0, 3 \pmod{6}
f(u_i) = 2, \quad i \equiv 4, 5 \pmod{6}

**Case (ii):** when n \equiv 1 \pmod{6}

f(u_{2n-3}) = 2
f(u_{2n-2}) = 1
f(u_{2n-1}) = 3

For 1 \leq i \leq 2n-4

f(u_i) = 1, \quad i \equiv 0, 1 \pmod{6}
f(u_i) = 3, \quad i \equiv 2, 5 \pmod{6}
\[ f(u_i) = 2, \quad i \equiv 3, 4 \pmod{6} \]

**Case (iii):** when \( n \equiv 2 \pmod{6} \)
\[ f(u_{2n-2}) = 2. \]
Labeling of the vertices \( u_i, 1 \leq i \leq 2n-1 \) (i \( \neq 2n - 2 \)) are same as in case (i).

**Case (iv):** when \( n \equiv 3 \pmod{6} \)
\[ f(u_{2n-1}) = 2. \]
Labeling of the vertices \( u_i, 1 \leq i \leq 2n-2 \) are same as in case (i).

**Case (v):** when \( n \equiv 4 \pmod{6} \)
Labeling of the vertices \( u_i, 1 \leq i \leq n-1 \) are same as in case (i) and \( f(u_n) = 3. \)
Labeling of the vertices \( u_i, n+1 \leq i \leq 2n-1 \) is given below.

For \( 1 \leq i \leq 2n-1 \)
\[ f(u_i) = 2, \quad i \equiv 1, 2 \pmod{6} \]
\[ f(u_i) = 3, \quad i \equiv 0, 3 \pmod{6} \]
\[ f(u_i) = 1, \quad i \equiv 4, 5 \pmod{6} \]

**Case (vi):** when \( n \equiv 5 \pmod{6} \)
\[ f(u_{2n-3}) = 3, \quad f(u_{2n-2}) = 1, \quad f(u_{2n-1}) = 2. \]
Labeling of the vertices \( u_i, 1 \leq i \leq 2n-4 \) are same as in case (i).

<table>
<thead>
<tr>
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**Table 4**
4. Conclusion
In this paper, a quotient-3 cordial for some standard graphs has been found. The quotient-3 cordial labeling of some more graphs and graph families shall be explored in future.

References